Worksheet Part 1: Lissajous Figures

The 3d pendulum follows trajectories given by this set of equations:

$x(t) = A * \sin(\omega_1 * t + \phi)$

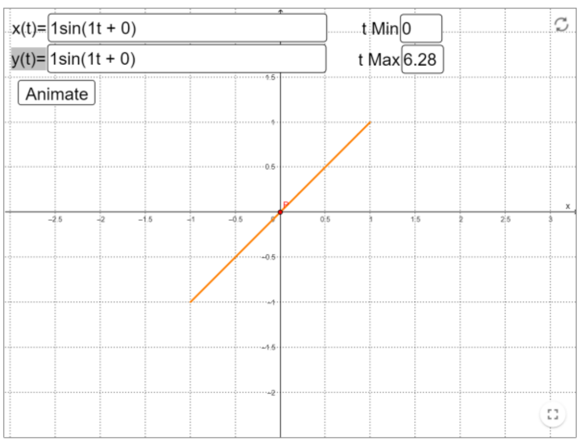
$y(t) = A * \sin(\omega_2 * t)$

Where ω_1 and ω_2 are the angular frequencies of the pendulum in each direction, t is time, and ϕ is the phase difference between the swings in the two directions. Don't worry to much about phase for now, we will focus on frequency. As a reminder, $\omega = \sqrt{\frac{g}{l}}$, and this relates to the period as $T = \frac{2\pi}{\omega}$.

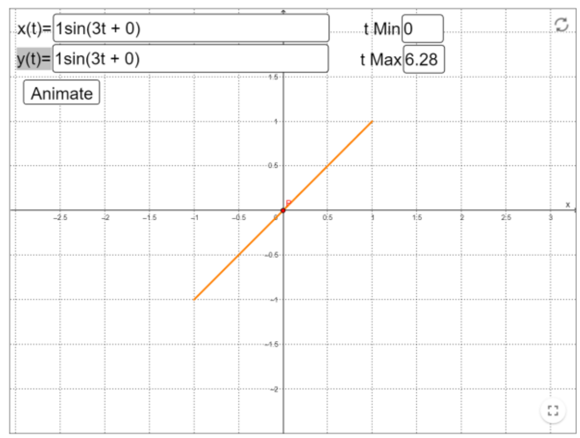
Since A just serves to scale the size of the trajectory (do you see why?) but doesn't change its shape, you can set A = 1 for the entire worksheet.

For part 1 of this worksheet, go to https://www.geogebra.org/m/yJNhQMQa

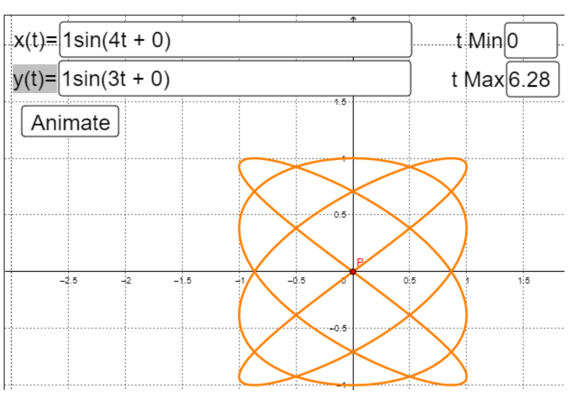
1. Enter the equations for x(t) and y(t). Set ϕ=0 and ω_1 = 1 = ω_2 . Animate the graph and draw the resulting shape below.



2. Now keep ω_1 = ω_2 but try other values. Do you still get the same shape?

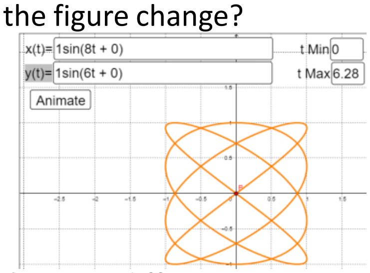


Now try the same thing but with $\omega_1 = 3$ and $\omega_2 = 4$ and draw the result.



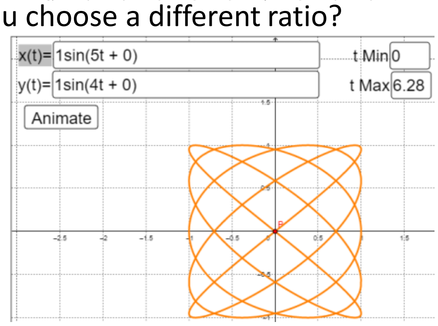
Try other values for ω_2 & ω_1 but still keeping the ratio $\frac{\omega_2}{\omega_1} \propto \frac{3}{4}$. Does the figure change?

It stays the same.



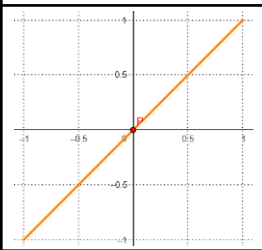
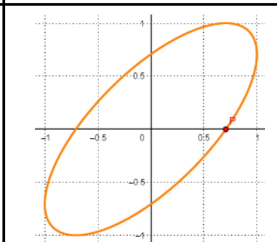
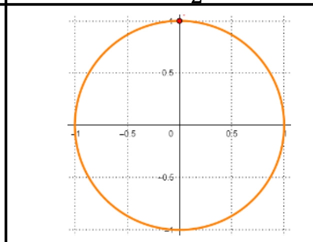
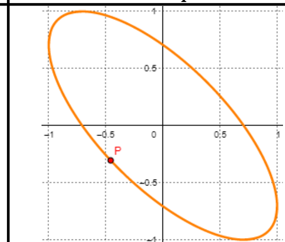
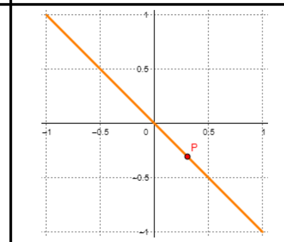
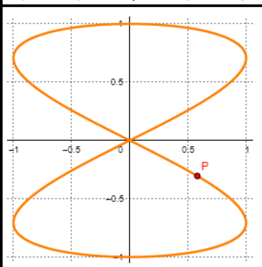
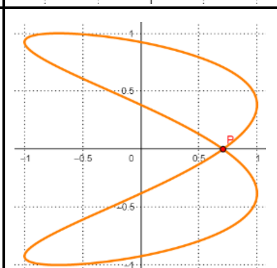
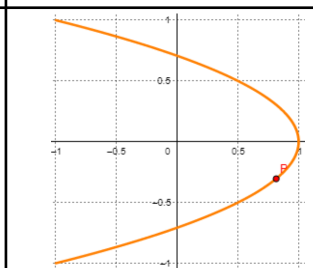
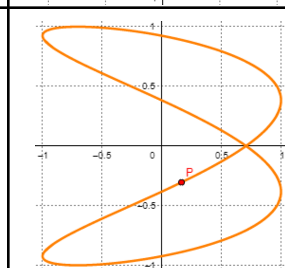
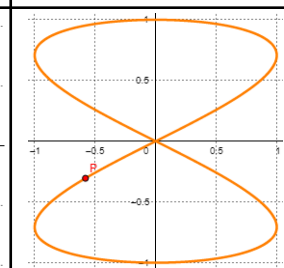
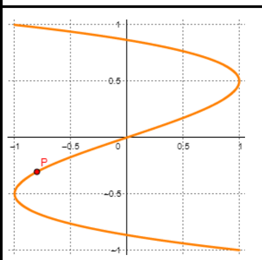
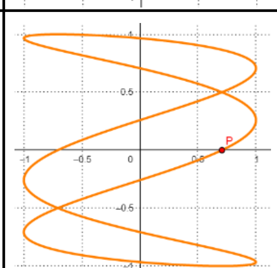
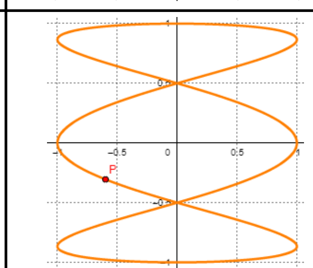
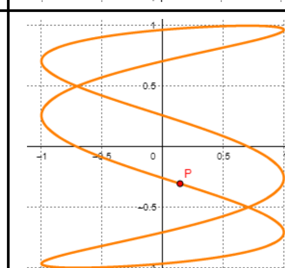
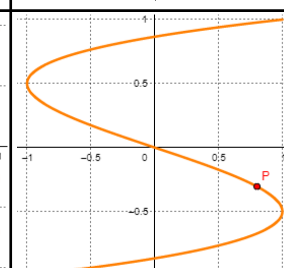
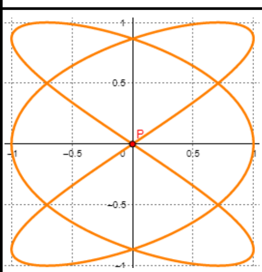
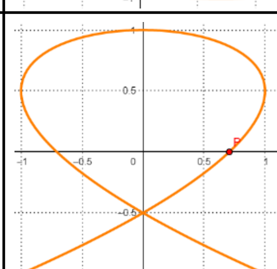
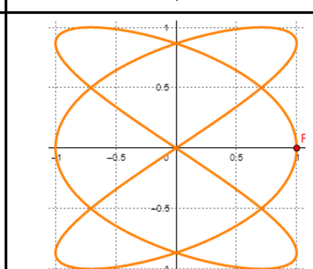
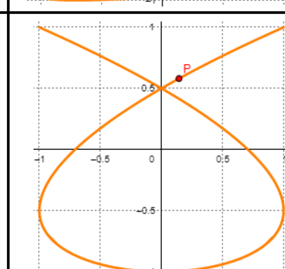
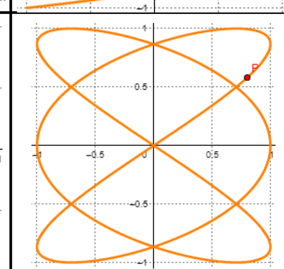
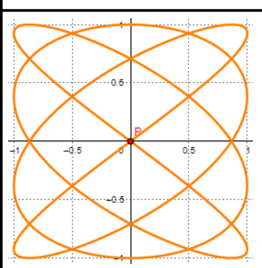
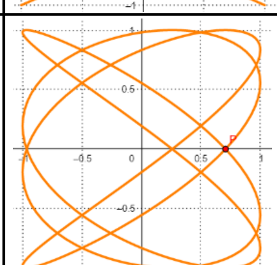
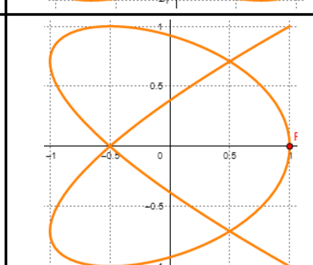
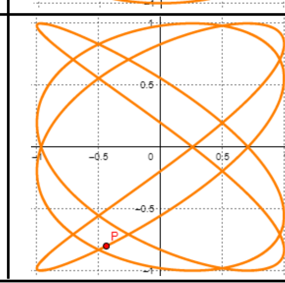
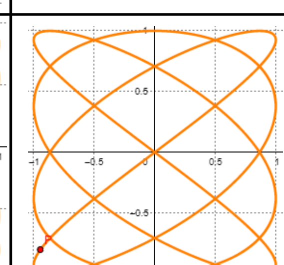
What about if you choose a different ratio?

It changes.



By plugging in the appropriate values, graphing the trajectory and sketching the result, fill in the table below. This kind of table is called a Lissajous table, and the figures you can make with these trajectories are called Lissajous figures.

If you don't have time to fill in all of it that's ok, prioritize the $\phi = 0$ column.

$\frac{\omega_2}{\omega_1} \propto$	$\phi = 0$	$\phi = \frac{\pi}{4}$	$\phi = \frac{\pi}{2}$	$\phi = \frac{3\pi}{4}$	$\phi = \pi$
1					
1/2					
1/3					
2/3					
3/4					

Looking at your table above try and answer the following questions. Explain your logic. It's ok if you don't have definite answers.

How can we predict the overall shape of the figure?

The overall shape of the figure is determined by the frequency ratio.

Can we predict how many loops a figure has? If so how?

Although some loops are squished into a single line sometimes, the frequency ratio also determines the number of loops: when $\frac{\omega_2}{\omega_1}$ is a ratio of integers, then ω_2 gives the number of loops across the top/bottom of a figure, and ω_1 gives the number of loops along each side.

Does the phase affect the overall shape of the figure? Describe how it changes the figure.

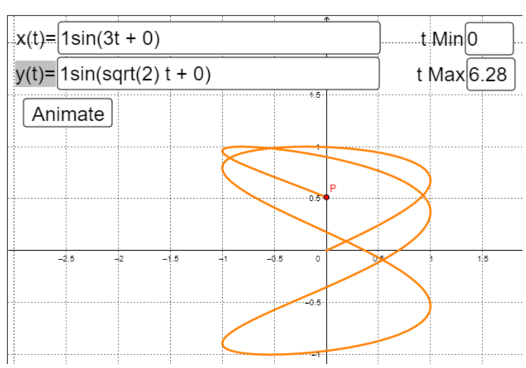
The phase just affects how squished/distorted the figure is.

One notices that as the phase changes from 0 to π the image gets flipped with mirror symmetry about the vertical line through it's center.

This could also be described as a gradual rotation about an axis in the plane of the page running through the center of the page. Distortion would then result from the projection back onto the plane of the page.

Bonus question (requires being more creative in the frequencies you try): What makes a figure bite its tail (if you imagine the figure is made of string there are no lose ends)?

The frequency ratio must be rational. Throwing in a $\sqrt{2}$, for example, gives lose ends right away:



Worksheet Part 2: String Lengths

I want to use the demo pendulum to make the beautiful figure with a ¾ frequency ratio in the Lissajous table. Using , $\omega = \sqrt{\frac{g}{l}}$ for both ω_2 & ω_1, find an equation for $\frac{\omega_2}{\omega_1}$ in terms of l_1 and l_2.

The desired Frequency Ratio is: $\frac{\omega_2}{\omega_1} \propto \frac{3}{4}$

Using $\omega = \sqrt{\frac{g}{l}}$, we know that, $\omega_1 = \sqrt{\frac{g}{l_1}}$ and $\omega_2 = \sqrt{\frac{g}{l_2}}$

So $\frac{\omega_1}{\omega_2} = \sqrt{\frac{l_2 g}{g\, l_1}} = \sqrt{\frac{l_2}{l_1}}$,

Using the equation you found above, what lengths (in meters) should I measure out for l_1 and l_2? Assume that the entire length from the bar the pendulum is hanging from to the bob is 1.5m.

Rearanging the result above and substituting in $\frac{\omega_2}{\omega_1} \propto \frac{3}{4}$ gives,

$$\frac{l_2}{l_1} = \left(\frac{\omega_1}{\omega_2}\right)^2 \propto \left(\frac{3}{4}\right)^2 \propto \frac{9}{16}.$$

Taking $l_1 = 1.5$m,

then, $l_2 = \frac{9}{16} * 1.5\text{m} \approx 84.375\text{cm}$.